SELF-SIMILAR CONVECTION NEAR A THERMAL QUADRUPOLE

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The self-similar solution is constructed for the Boussinesq equations describing free convection near a differentially heated local section on a horizontal plane.

Conically symmetric flows are an extensive and important class of self-similar solutions of the equations of motion of a viscous incompressible fluid. Among them are the known Landau [1] and Squire [2] analytic solutions as well as solutions of problems of a waterspout, of swirling jets possessing quite nontrivial properties [3]. A distinctive feature of the class is the inversely proportional dependence of the velocity on the distance to the origin $v \sim 1/R$. Consequently, viscous and convective momentum transfer generate fluxes of identical order of magnitude in generally the whole flow domain. The interaction, or more accurately, counteraction results occasionally in paradoxical effects [4].

If an analogous class of self-similar solutions is examined for the thermogravitational convection equations in the Boussinesq approximation, then it is easy to see that the Archimedes force should be proportional to $1/R^3$. The problem of a point source of heat is seemingly simplest. Precisely such a model was used in a number of papers [5-8] to analyze the influence of the buoyancy forces on jet viscous fluid flows. However, in the problems considered the Archimedes force and temperature are proportional to $1/R$. Consequently, the self-similar solutions were obtained within the framework of the boundary layer approximation and under the assumption of smallness of the buoyancy effects.

Another case is examined here, when the point singularity of the temperature field is a quadrupole, which assures the necessary dependence $T \sim 1/R^3$. The total heat flux from the singular point is zero here while the temperature distribution on a sphere of given radius is sign-variable in nature. Despite a certain exoticness of such a distribution, tied to the self-similarity requirement, such a formulation is the simplest scheme for real situations. It can be located on the horizontal plane of a hemisphere on which a differential temperature distribution is given. If the apex is hot and the base is cold, then such a situation will provisionally be called a "volcano," while the inverse is an "iceberg." As the hemisphere radius tends to zero, and under an additional requirement of a zero total heat flux, the problem under consideration will be obtained.

Thus, on a plane $\Theta = \pi/2$ let a temperature distribution be given according to the law $T = T_\infty + \gamma/R^3$. Here and henceforth, a spherical (R, Θ, φ) coordinate system is utilized. The axisymmetric case is examined when the velocity field $(v_R, v_\Theta, 0)$ and temperature are independent of the azimuthal angle φ, here Θ is the angle between the radius-vector and the axis of symmetry. Satisfaction of the attachment conditions is required on the plane.

The formulated boundary value problem allows for a two-parameter class of solutions dependent on the Grasshof $Gr = \beta\gamma g/\nu^2$ and Prandtl $Pr = \nu/\chi$ criteria. Because of the absence of a characteristic length scale, absolute self-similarity holds [9] and the following representation is valid:

$$v_R = -\nu y'(x)/R; \quad v_\Theta = -\nu y(x)/(R\sin\Theta); \quad x = \cos\Theta;$$
$$p = p_\infty + \rho\nu^2 g(x)/R^2; \quad T = T_\infty + \gamma\vartheta(x)/R^3.$$

Here and henceforth the prime denotes differentiation, and without loss of generality, the values of p_∞, T_∞ can be considered zero.

After substituting these expressions into the free convection equation [1] and simple manipulation, the problem is reduced to a system of ordinary differential equations

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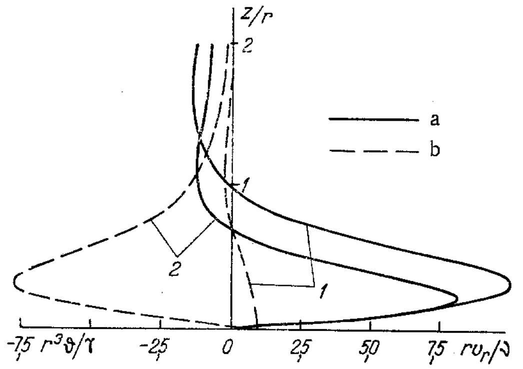

Fig. 1. Distribution of the horizontal velocity (a) and temperature (b) with height: 1) Pr = 0, Gr = 200; 2) Pr = 0.7, Gr = 35.

$$(1-x^2)\,y^{\mathrm{IV}} - 4xy - (y^2/2)''' = \mathrm{Gr}\,(x\vartheta' + 3\vartheta), \tag{1}$$

$$(1-x^2)\,\vartheta'' - 2x\vartheta' + 6\vartheta = \mathrm{Pr}\,(y\vartheta' + 3y'\vartheta) \tag{2}$$

with the boundary conditions

$$y(0) = y'(0) = 0;\quad \vartheta(0) = 1;\quad y(1) = 0;$$
$$|y'(1)| < \infty;\quad \vartheta'(1) = 3\,[1 - \mathrm{Pr}\,y'(1)/2]\,\vartheta(1).$$

The last two conditions are associated with the requirement of analyticity of the velocity and temperature fields on the axis of symmetry.

Following the methodology in [3], it is convenient to introduce the function F(x) such that

$$F''' = \mathrm{Gr}\,(x\vartheta' + 3\vartheta);\quad F(1) = F'(1) = F''(1) = 0. \tag{3}$$

Then (1) is integrated thrice and with the boundary conditions taken into account for x = 1 reduces to the form

$$(1-x^2)\,y' + 2xy - y^2/2 = F(x) - C\,(1-x)^2. \tag{4}$$

By virtue of the boundary conditions on a plane the constant of integration C is connected to F(x) by the relationship C = F(0).

The following representation is valid for the function F(x)

$$F(x) = -\frac{\mathrm{Gr}}{2}\int\limits_x^1 (x-t)^2\,[t\vartheta'(t) + 3\vartheta(t)]\,dt,$$

from which

$$F(0) = -\frac{\mathrm{Gr}}{2}\int\limits_0^1 [t^3\vartheta(t)]'\,dt = -\frac{\mathrm{Gr}}{2}\,\vartheta(1).$$

Consequently, it is expedient to integrate the system (2)-(4) between x = 1 and x = 0 by giving trial values $\vartheta(1)$ and $y'(1)$ and thereby reducing the boundary value problem to a Cauchy problem for the numerical analysis. The quantity $y'(1)$ is not determined by (4) but is a free parameter because x = 1 is a singular point of (4). The value ϑ'' is found separately by means of differentiating (2) and (4) and utilizing the analyticity conditions. Consequently, after simple calculations

$$\vartheta''(1) = [1 - \mathrm{Pr}y'(1)]\,\vartheta'(1) - \frac{3}{4}\,\mathrm{Pr}\,\vartheta(1)\,\{C + y'(1)\,[1 - y'(1)/2]\}.$$

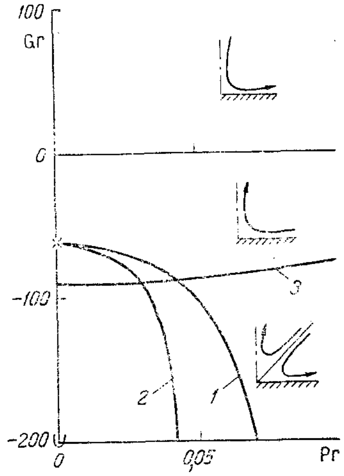

Fig. 2

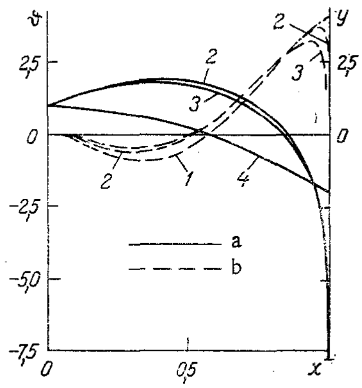

Fig. 3

Fig. 2. Convection regime pattern. For curves 1 and 2, respectively, $y'(1) = -460.5; -3000$. Curve 3 separates the domains of one and two-cell regimes. ×, U are the boundaries for existence of the regular and logarithmically singular solutions with respect to the temperature for Pr = 0.

Fig. 3. Nature of the passage to the limit as Pr → 0 for the velocity (b) and the temperature (a), Gr = -150. Curves 1, 2, and 3 correspond to Pr = 0, 0.04, 0.063 and 4 to $\vartheta = 1-3x^2$.

The quantity $y'(1)$ must be selected such that $y(0) = 0$ will be obtained as a result of integration. The condition $y'(0) = 0$ is here satisfied automatically. Then by renormalizing $\vartheta(x)$ so that $\vartheta(0) = 1$, we obtain the solution of the initial problem.

In the Pr = 0 case, (2) is integrated analytically: $\vartheta(x) = 1-3x^2$. The temperature on the axis is twice the temperature on the plane and has the opposite sign. For Gr > 0 we have an "iceberg" type situation, and for Gr < 0 a "volcano." Applying the substitution $y = -2(1-x^2)U'/U$ and taking account of the temperature distribution obtained, we easily arrive at the equation

$$U'' - \frac{\mathrm{Gr}\,x}{8}\,U = 0,\ U(0) = 1,\ U'(0) = 0. \tag{5}$$

A solution of (5) is the Airey function [10] that has a representation of an everywhere convergent series

$$U = \sum_{n=0}^{\infty} a_n t^n;\ t = \mathrm{Gr}\,x^3/48;$$

$$a_0 = 1;\ a_n = 2a_{n-1}/[n(3n-1)],\ n = 1,\ 2,\ \ldots$$

The function $U(x)$ should not have zeroes in the interval $0 \leqslant x < 1$ for $y(x)$ to be regular. For Gr > 0 the function $U(x)$ is positive for all $x \geqslant 0$; consequently, the iceberg problem is solvable for any quadrupole intensity. Since $U'(x) > 0$ for $x > 0$ then within the interval $y(x) < 0$. This means that the flux is directed downward along the axis and later spreads onto the plane from the origin.

Let us consider the motion for Gr >> 1 for which we return to (4) which has the following form in case Pr = 0

$$(1-x^2)\,y' + 2xy - y^2/2 = -\mathrm{Gr}\,x\,(1-x^2)^2/4.$$

For Gr >> 1 by neglecting linear terms in the left side we obtain the potential equation

$$y_{\mathrm{p}} = -(1-x^2)\sqrt{\mathrm{Gr}\,x/2}.$$

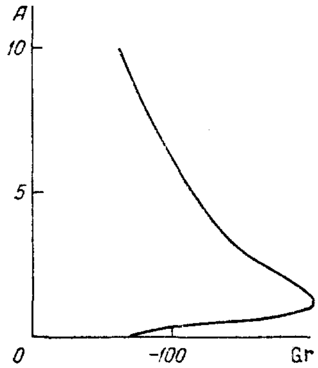

Fig. 4. Dependence of the coefficient for the logarithmically singular solution on the Grasshoff number for Pr = 0.

The derivative y'_p becomes infinite for x = 0, which corresponds to an infinite velocity on the plane. Taking account of the influence of viscosity a boundary layer occurs near the plane.

The equation linearized with respect to y has the solution

$$y_\ell = -\mathrm{Gr}\, x^2 (1 - x^2)/8,$$

which is applicable for small Grasshoff numbers in the whole domain and in a small neighborhood of the wall for large numbers.

There follows from the requirement of adjointness of the solutions $y_\ell(x)$ and $y_p(x)$ that the thickness of the near-wall boundary layer is $\sim \mathrm{Gr}^{-1/3}$ while the maximal value of the longitudinal velocity is $\sim \mathrm{Gr}^{2/3}$. Therefore, as the Grasshoff number increases a powerful near-wall jet is formed.

For Pr > 0 the velocity distribution remains qualitatively the same as for Pr = 0. However, the predominance of the convective mechanism of heat transfer for large Grasshoff numbers results in the fact that the domain of positive temperatures diminishes to the thin near-wall layer of width ~1/(PrGr) (Fig. 1). The results of computations are represented in cylindrical (z, r, φ) coordinates for greater clearness

$$v_r = -\frac{\nu}{r}[(1 - x^2)\, y' + xy]; \; r = R \sin \Theta; \; z = R \cos \Theta.$$

Summarizing the analysis of the case Gr > 0 it can be said that a strong cold wind blows from the "iceberg", as should generally be expected.

The "volcano" problem is more abundant in surprises. Again we first set Pr = 0. Since Gr < 0 the solutions (5) are oscillatory in nature for positive x. In order for the solution y(x) not to have poles the roots of U(x) should be outside the interval (0, 1). The smallest zero in absolute value for the function U(t) equals $t_1 \approx -1.306$ while $x_1 = (48t_1/\mathrm{Gr})^{1/3}$. As |Gr| grows, the quantity x_1 diminshes and becomes equal to one for $\mathrm{Gr} = \mathrm{Gr}_* = 48t_1 \approx -62.7$. The zeroes of U(x) and U'(x) alternate, as follows from (5); consequently if $|\mathrm{Gr}| < |\mathrm{Gr}_*|$, then in the interval (0, 1), U(x) > 0, U'(x) < 0, i.e., y(x) > 0. This means that the flow is ascending, along the plane it is directed towards the origin, and later upward along the axis of symmetry. As x_1 approaches one $y'(1) \to -\infty$. Therefore, a strong jet is developed near the axis.

For $x_1 = 1$ the roots and poles of the function y(x) merge at the point x = 1. The value of y(1) becomes different from zero but finite: y(1) = 4. In other words, in the critical situation the jet momentum and the velocity on the axis become infinite but the ejection capacity of the jet remains finite and a sink with the volume mass flow rate 8πν per unit length is formed on the axis. For $|\mathrm{Gr}| > |\mathrm{Gr}_*|$ the solution of the boundary value problem posed ceases to exist. An analogous paradox was first detected and investigated in [4].

A small parameter $\varepsilon = -1/y'(1)$ and an "interior" coordinate $\eta = (1-x)/\varepsilon$ can be introduced in the near-critical situation. Then the principal term of the interior asymptotic expansion in ε is the solution

$$y^* = 4\eta/(4 + \eta),$$

which agrees with the Schlichting solution for a circular jet in the boundary layer approximation [1].

For Pr > 0 there is no crisis, the solution exists even for $|Gr| \gg |Gr_*|$ (computations were performed up to $Gr = -10^3$). The results represented in Fig. 2 indicate that for small values of $|Gr|$ (above curve 3 in Fig. 2) the flow regime is one-celled while for large $|Gr|$ it is two-celled (see the diagram in Fig. 2). As Pr grows the line of separation asymptotically has the nature $Gr \approx -16/Pr$. The flow in the two-cell regime is directed to the origin along a conical surface on which the temperature is negative and then it separates into two branches. One part of the flow spreads along the plane from the origin, while the other forms a near-axis jet beating upward. Therefore. for sufficiently large Grasshoff numbers and in the "volcano" situation a stong cold wind blows on the surface while the heat of the "volcano" entrains the jet upward.

The question arises as to what occurs as $Pr \to 0$. When $|Gr| < |Gr_*|$ the solution goes over into that obtained for Pr = 0 in a continuous manner. The postcritical transition is illustrated by curves 1 and 2 in Fig. 2 on which $y'(1)$ = const. These curves have the semiaxis $Gr < Gr_*$, Pr = 0 as limit when $y'(1) \to -\infty$. The maximal value of y(x) tends to 4 here and the location of the maximum to x = 1. Physically this means that as the Prandtl number diminishes near the axis a Schlichting jet is formed and accentuated. In this respect the same occurs also for Pr = 0, $Gr \to Gr_*$. But the temperature distribution does not tend to the solution considered above for Pr = 0 (Fig. 3).

In (2) the right side tends to zero for any $x \neq 1$ as $Pr \to 0$. A singular case is x = 1 since $y'(1) \to -\infty$, where $y'(x)$ acquires a singularity of delta function type in the limit, for which the coefficient is asymptotically independent of the Prandtl number and equals 4. The $y'(x)$ in (2) is multiplied by $3Pr\vartheta(x)$. If $\vartheta(1)$ were to remain bounded in the limit, then the coefficient of the delta-function would vanish and it would not influence the nature of the solution. It remains to admit that $\vartheta(1) \to -\infty$ as $Pr \to 0$. The numerical computations indicate this favorably indeed. Therefore, in order to find the limit solution one should not limit oneself to the analytic solution of (2) for Pr = 0. The general solution has the form

$$\vartheta = A_1(1-3x^2) + A\left[6x + (1-3x^2)\ln\frac{1+x}{1-x}\right].$$

By virtue of the normalization conditions $A_1 = 1$. The coefficient A is to be determined for the solution with the logarithmic singularity. We find as a result of integrating (3)

$$F(x) = \frac{Gr\,(1-x^2)^2}{4}\left[A\left(2 - x\ln\frac{1+x}{1-x}\right) - x\right],$$
$$C = F(0) = A\,Gr/2.$$

Substituting these expressions into (4) we solve the boundary value problem for it: $y(1) = 4$, $y(0) = 0$. As a result of differentiating (4) we determine $y'(1) = 2$. (This value of the derivative corresponds to the "outer" expansion in ε). Consequently, for x = 1 there is all the necessary information for the solution of the Cauchy problem. The connection between the parameters A and Gr is found from the necessity to satisfy the condition y(0) = 0 (Fig. 4). The lower branch of the dependence A(Gr) corresponds to the limit solutions as $Pr \to 0$. In Fig. 3 where such a passage to the limit is shown, the temperature distribution for Pr = 0.04 is already graphically indistinguishable from the limit, convergence in y(x) is slower, "smoothing" of the logarithmic singularity in $\vartheta(x)$ noticeably influences F(x) and C.

It is seen from Fig. 4 that for sufficiently large Grasshoff numbers ($|Gr| > 202$) no solution with a logarithmic singularity exists if the Prandtl number equals zero. The passage to the limit $Pr \to 0$ for $|Gr| > 202$ results in a solution with a stronger singularity than the logarithmic in the temperature distribution.

The upper branch of the dependence A(Gr) in combination with the lower one bounds the domain of existence of solutions of another problem for Pr = 0 when a logarithmic singularity is given a priori for the temperature field on the axis of symmetry in addition to the quadrupole at the origin. Solutions satisfying the conditions $y(1) = 0$, $|y'(1)| < \infty$ exist in the domain lying to the left of the curve presented in Fig. 4. They are also separated into two-cell and one-cell. Friction on the plane is determined by the quantity $y''(0) = Gr\,(A - 1/4)$.

Consequently, for $A \leq \frac{1}{4}$ the flow is purely ascending while separation occurs as the quantity A exceeds the value $\frac{1}{4}$ and a return flow occurs near the plane.

What are the physical reasons for the occurrence of singularities in the temperature distribution in the quadrupole problem as the Prandtl number tends to zero? Seemingly, the relative value of heat convection drops as Pr diminishes and diffusion should predominate, which usually smoothes all singularities. The paradoxicality of the situation in this case is explained by the fact that the velocity near the axis becomes infinite for sufficiently high Grasshoff numbers as Pr diminishes. Consequently, the convective mechanism remains governing near the axis. Development of the singularity is assured by the positive feedback between the momentum and heat transfer. An increase in the jet velocity raises the ejection and its associated convective heat transfer to the axis. Heat accumulation at the axis results in the growth of the Archimedes force there and, therefore, of the jet momentum.

NOTATION

(R, Θ, φ), (r, z, φ), spherical and cylindrical coordinates; v, v_R, v_Θ, v_r, velocity and its appropriate components; T, temperature; ρ, density; p, pressure; ν, kinematic viscosity; χ, thermal diffusivity; Gr, Pr, Grasshoff and Prandtl criteria; x, t, η, auxiliary dimensionless arguments; y, ϑ, F, U, auxiliary dimensionless functions; C, A, A_1, numerical constants; γ, a factor in the temperature distribution at the wall [$K \cdot m^3$]. Subscripts: ∞, parameters at infinity; *(subscript), critical value, and *(superscript), internal decomposition.

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